Measuring the momentum of light

Af

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In physics, momentum is one of the most fundamental properties. The law of conservation of momentum is universal. According to the special theory of relativity, light has momentum despite the fact that a photon has no mass. This claim is very surprising, for how can something without mass have momentum? From Newton's second law of motion, we know that something that has momentum can exert a force. So can light exert a force? If so, can we measure it?

This article describes a first-year physics project to measure the force exerted by light and thus show that light does indeed have momentum, conducted by Kjeld Bak, Christian Prag and myself in the spring of 2008, under the guidance of Ian Bearden and Mogens Levinsen.

We constructed a torsion pendulum in a vacuum chamber and fired a powerful laser at a mirror attached to the pendulum. The pendulum rotated away from the laser light and we could measure the force of the light by the angle of deflection. By using a second laser to measure the angle of deflection and a photosensor attached to a computer, we could see the effect of light on the pendulum in real-time, and were able to record precise data about its motion.

The results were very encouraging. We were able to both visibly see the effect of light on the pendulum and measure a force in the order of 10^{-9}N. The measured force was within 6 % of that predicted by theory.

1 Theory of the momentum of light and its force

In order for the law of conservation of momentum to hold for inertial systems moving at a significant fraction of the speed of light, the special theory of relativity added the Lorentz-factor γ to the classical definition of momentum.

$$\vec{p} = \gamma m \vec{v} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1}$$

As the speed of a particle v approaches the speed of light c, the momentum approaches infinity. For photons ($v = c$), although the Lorentz-factor is infinite, the mass is also zero. So what then is the momentum of a photon? The answer comes from the theory of special relativity and the equation for the total energy of a particle:

$$E^2 = (mc^2)^2 + (pc)^2 \tag{2}$$

For a photon $m = 0$ and we get $E = pc$ or $p = E/c$, which written as a vector becomes:

$$\vec{p} = \frac{E}{c}\vec{n} \tag{3}$$

Where $\vec{n}$ is a unit vector that points in the direction of the photon's movement. The force on the photon is the time-derivative of its momentum and for a beam of light with a given power is:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{E}{c}\frac{d\vec{n}}{dt} = \frac{E}{c}\frac{\Delta\vec{n}}{\Delta t} = \frac{Power}{c}\Delta\vec{n} \tag{4}$$

Where $\vec{n}$ is the change in direction of the beam of light.

In our experiment, we shall measure the force that light exerts on a mirror. In a collision between a photon and a mirror, the force exerted on the mirror is equal to the force exerted on the photon, but in the opposite direction, according to Newton's third law of motion. In such a collision, the photon can be reflected by, absorbed by or transmitted through the mirror. For reflection $|\vec{n}| = 2$, for absorption $|\vec{n}| = 1$ and for transmission $|\vec{n}| = 0$. If we define the fraction of photons in a beam of light that are reflected, absorbed or transmitted as r_s, a_s and t_s, we get the following expression for the magnitude of the force exerted on a mirror by a beam of light:

$$F = \frac{Power}{c}(2r_s + a_s) \tag{5}$$

In the case where t_s is zero, $a_s = 1 - r_s$ and the force on the mirror becomes:

$$F = \frac{Power}{c}(1 + r_s) \tag{6}$$

For a typical 1mW laser pointer, the force on a perfect mirror ($r_s = 1$) will be ca. $6 \cdot 10^{-12}N$. This is very little force!

1.1 Other force effects

Light exerts not only a direct force on the mirror, but absorbed energy heats the mirror, and this can result in an additional force on the pendulum. The effect is known as the light-mill effect, and was first seen by Sir William Crookes in 1873. The device most often used to show students the light-mill effect resembles a clear glass light bulb, containing a vacuum and several blades, painted black on one side and white on the other, which are free to rotate. When exposed to light, the blades rotate, but do not demonstrate the force of light, but rather the force of small air currents, resulting from convection.

The light-mill effect exerts a maximum force at a pressure of around 10^{-2} torr (10^{-5} atm), which decreases with reduced pressure and eventually disappears by 10^{-6} torr (10^{-9} atm). In order to reduce light-mill effects in our experiment, we placed the pendulum in a high vacuum with a pressure of around 10^{-5} torr (10^{-8} atm).

Another effect that can exert a force on the pendulum is out-gassing, where small pockets of gas molecules in the mirror's surface are ejected away from the surface, because of low pressure or heating. The effect is generally short-lived; occurring immediately after a material is placed in a vacuum. Since the pendulum was maintained in a vacuum for several weeks, we do not expect this effect to influence the results.

Static electric charges can also exert a force on the pendulum. We avoided this by grounding the pendulum and the vacuum chamber.

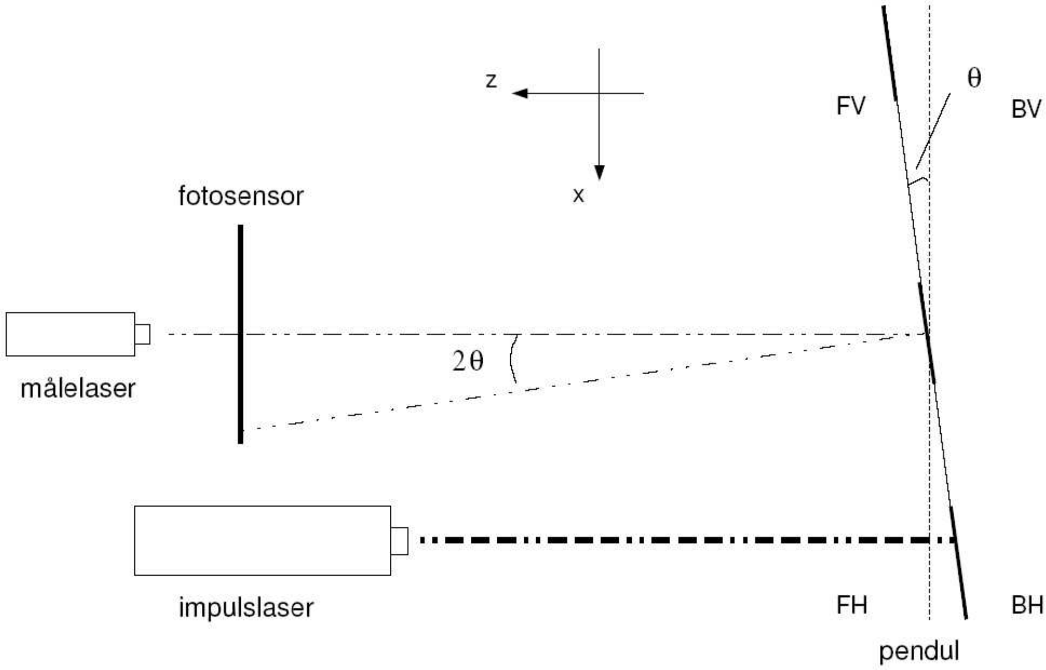
2 Experiment setup

We used a torsion pendulum in a manner similar to that used by Cavendish to measure the force of gravity, and used by E.F. Nichols and G.F. Hall in 1901 to measure the pressure of light (Nichols & Hull (1903)). The experiments by Nichols and Hull were among the first to succeed in measuring the force of light with reasonable accuracy. In our case, though, we used lasers, a modern day vacuum pump, and a computer for data collection.

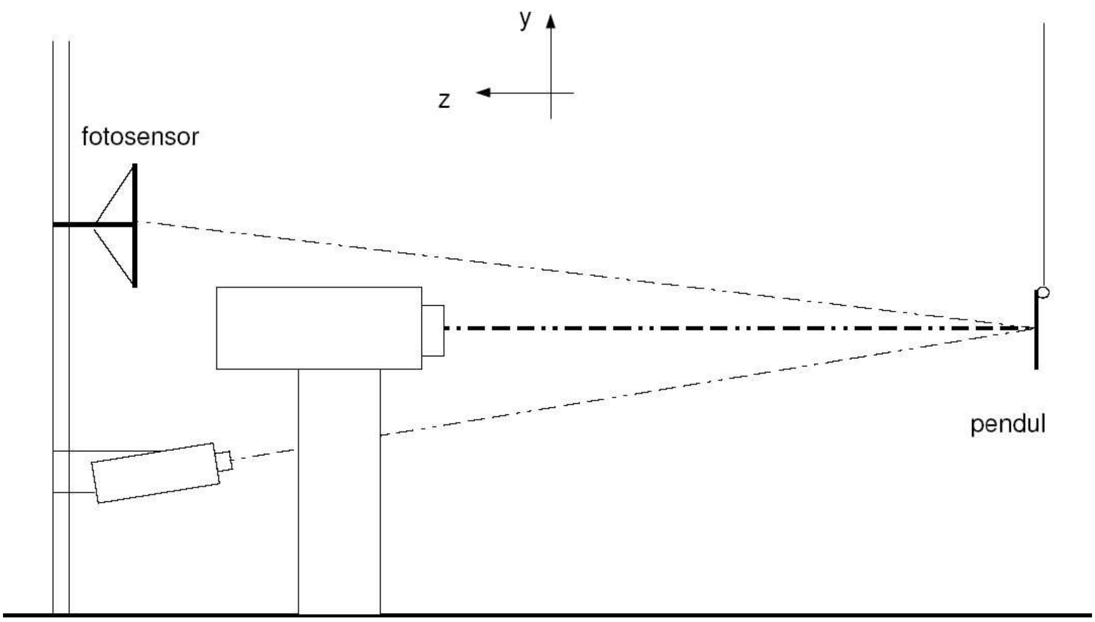
The simple purpose of the experiment is to aim a laser beam at a mirror attached to a torsion pendulum, and to measure if the amount it rotates is in agreement with the force that we expect the light beam of the laser to have. Our experiment's setup is sketched in figure 1 and figure 2.

The torsion pendulum (pendul) consists of a balanced horizontal bar (see figure 3), upon which three small mirrors are attached: one at each end, for providing a torque from the incident light, and one in the middle, for measuring the angle of rotation. The horizontal bar is made of steel, and ca. 140 mm long. It is hung from its center by a thin thread of gold-plated wolfram. The mirrors are made of thin glass, vacuum-coated with aluminum, and ca. 20 mm square.

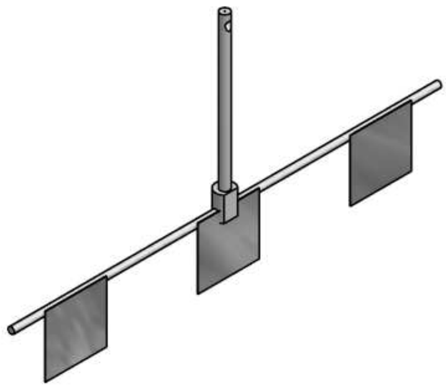
We used two lasers: one to exert a force on one of the end-mirrors - the source-laser (impulslaser), and one to measure the rotation of the pendulum - the measurement-laser (målelaser). Our light source was a powerful 500 mW laser, with a wavelength of 726 nm, which is on the border of the infrared spectrum. We were fortunate to be able to loan the laser from Torsana Laser Technologies. We used a laser power meter to measure the source-laser's power before and after each experiment. The measurement-laser was a standard 5 mW green laser pointer. We connected the laser pointer to a DC power supply, instead of using batteries,



Figur 1: Experiment setup seen from above



Figur 2: Experiment setup seen from the side



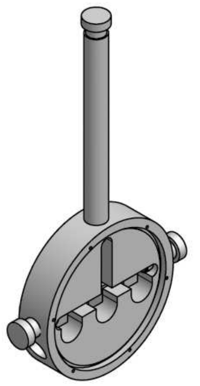
Figur 3: The torsion pendulum

and cooled it with a fan, to avoid variation in its power output.

A photosensor (fotosensor) connected to a computer was used to record the position of the measurement-laser's reflection from the center-mirror. The photosensor was a small 10x10 mm plate, which gave the position of a laser spot on its surface, in both the horizontal and vertical directions, as a voltage difference. We wrote a program in LabVIEW to read data from the photosensor, and once calibrated, we used this data and the geometry of the setup, to determine the angle of deflection of the pendulum. Not only could we measure an angle of deflection smaller than the eye could see, but we could see the deflection in real-time on the computer screen as the pendulum reacted to the source-laser. As well, we could calculate the force applied without waiting for the pendulum to come to rest.

Since the force of light is so small, the pendulum must be very sensitive. Air can affect the results, either by dampening the pendulums motion, or by exerting a force from convection (the light-mill effect). Therefore, a vacuum chamber (see figure 4) was constructed by NBI's workshop to house the pendulum. The vacuum chamber is made of aluminum and covered on both ends by glass plates. Attached to the top is a pipe wherein is hung from a hook the thread suspending the pendulum. One can rotate the hook in order to zero the pendulum's position.

A vacuum pump was connected to the vacuum chamber and a pressure meter was used to read the pressure during each experiment. Unfortunately, our vacuum chamber was not completely tight, so we could not lower the pressure below 10^{-5} torr (10^{-8} atm). This was, however, still a



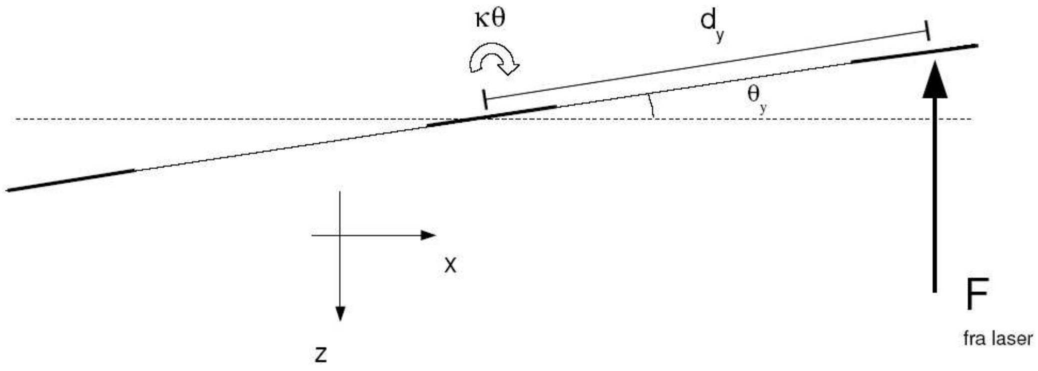
Figur 4: Vacuum chamber

good high vacuum and sufficient for performing the experiment.

3 Experiment theory

We define the pendulum's suspension point as the zero point of our coordinate system, with the three axes shown in figure 1 and figure 2. The y-axis points upwards parallel to the thread, the x-axis points to the right seen from the photosensor, and the z-axis points from the pendulum towards the photosensor.

The pendulum hangs from a hook so that it can rotate freely about the suspension point around all three axes. The primary rotation that we are interested in is the twisting rotation about the thread (around the y-axis), as this can most easily be used to measure the force of the incident light. The two secondary rotations, a back-and-forth swinging (around the x-axis) and a side-to-side swinging (around the z-axis), are less interesting. Since the force exerted by the source-laser is parallel to the z-axis, it cannot induce rotation about the z-axis. As well, rotation about the x-axis is very small when compared to the y-axis, since the force from light must compete with gravity. Around the y-axis, only torsion impedes the pendulums motion.In this article, I will only look at rotation around y-axis, i.e. around the thread suspending the pendulum. For those that are interested, the secondary axes are discussed in more detail in the original report.



Figur 5: Force diagram for the primary axis of rotation

3.1 Equation of motion

The forces for rotation of the pendulum about the y-axis are shown in figure 5. The motion can be described by using the rotational analog of Newton's second law for torque around the y-axis τ_y.

$$\sum \tau_y = F d_y cos(\theta_y) - \kappa \theta_y = I_y \ddot{\theta}_y \tag{7}$$

Where F is the force from the incident light, d_y is the distance of the light from the rotational axis (y-axis), κ is the torsion constant for the thread, θ_y is the angle from the equilibrium position, I_y is the moment of inertia of the pendulum about the y-axis, and $\ddot{\theta}_y$ is the angular acceleration. For a small angle $cos(\theta) \approx 1$ and the angular acceleration becomes:

$$\ddot{\theta}_y = -\frac{\kappa}{I_y}\theta_y + \frac{F d_y}{I_y} \tag{8}$$

Equation 8 is a second order non-homogenous differential equation. Let us begin by solving it for the simple case where the pendulum is not in motion and in equilibrium when the source light is turned-on at time $t = 0$. In this case, the equation has the following solution:

$$\theta_y(t) = \frac{F d_y}{I_y \omega_y^2}(1 - cos(\omega_y t)) \qquad \text{for} \quad t \geq 0 \tag{9}$$

$$\omega_y = \sqrt{\frac{\kappa}{I_y}} \tag{10}$$

he solution describes a simple harmonic oscillation about a central position, where ω_y is the angular frequency. The central position is the new equilibrium position and is given by:

$$\theta_{y,equilibrium} = \frac{Fd_y}{I_y\omega_y^2} = \frac{Fd_y}{\kappa} \tag{11}$$

Generally, the pendulum will already be oscillating prior to turning-on the source-light. For instance, the oscillations of the torsion pendulum used in this experiment were essentially undamped, and it would continue to oscillate for a very long time. In this case, in order to solve the differential equation, we need to know its amplitude and phase at time $t = 0$. If A is the amplitude and φ is the phase of oscillation for $t \leq 0$, the solution to the differential equation is:

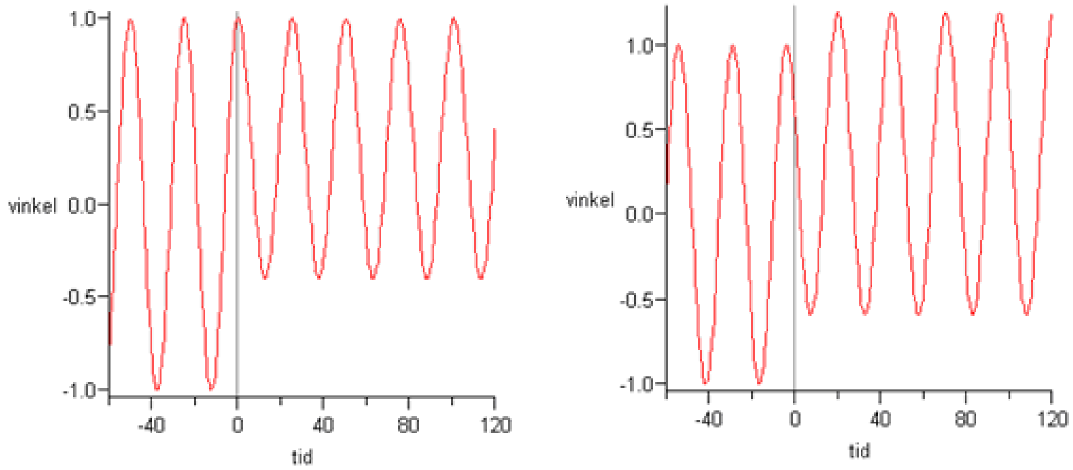
$$\theta_y(t) = Acos(\omega_y t + \varphi) \qquad \text{for} \quad t \leq 0 \tag{12}$$

$$\theta_y(t) = Acos(\omega_y t + \varphi) + \frac{Fd_y}{I_y\omega_y^2}(1 - cos(\omega_y t)) \qquad \text{for} \quad t \geq 0 \tag{13}$$

Note that eq. 12 is the general solution to the homogenous differential equation ($F = 0$), and eq. 9 is a particular solution to the non-homogenous differential equation. The total solution eq. 13 is the sum of eq. 12 and eq. 9 as defined by the superposition principle. The solution is shown in figure 6 for two different angular phases.

When the source-light is turned-off, the solution is of the same form, but the time range is swapped, that is eq. 13 for $t \leq 0$ and 12 for $t \geq 0$.

One can also find other forms of solutions to eq. 8, for example $\theta_y(t) = C_1 cos(\omega_y t + C_2) + C_3$. All solutions will describe a simple harmonic oscillation around an equilibrium position, where the equilibrium position



Figur 6: Pendulum motion when the incidence of the light occurs at the angular phases of 0 and $\pi/3$

changes instantaneously with the addition or removal of the source-light. The change is given by:

$$|\Delta\theta_{y,equilibrium}| = \frac{Fd_y}{I_y\omega_y^2} = \frac{Fd_y}{\kappa} \tag{14}$$

Given the change in equilibrium position and the angular frequency, it is possible to calculate how much force the light exerts. In the experiment, we measure the change in equilibrium position for oscillation around the y-axis, and thus find the force exerted by the light.

$$F = \frac{|\Delta\theta_{y,equilibrium} I_y\omega_y^2|}{d_y} \tag{15}$$

3.2 Theoretical force

Equation 6 describes the theoretical force for the reflection on a mirror. However, in our experiment, the source-light comes though the glass plate of the vacuum chamber. Therefore, we must add a factor t_g to represent the fraction of photons which are transmitted through the glass plate, when calculating the theoretical force.

$$F = \frac{\text{Effekt}}{c}(1 + r_s)t_g \tag{16}$$

3.3 Energy

A simple harmonic oscillator has both potential and kinetic energy. The pendulum swings between having all of its energy in kinetic form to having all of its energy as potential energy. Without external forces, the pendulums total energy will be constant and equal to the maximum potential or kinetic energy.

When the force from the light is added to or removed from the pendulum, it instantaneously changes the pendulums equilibrium position. This instantaneously changes the pendulums potential energy and thus its total energy, but does not change its kinetic energy. The total energy and thus the amplitude of the oscillation will be increased or decreased depending on when we add or remove the force from the light.

4 Experiment

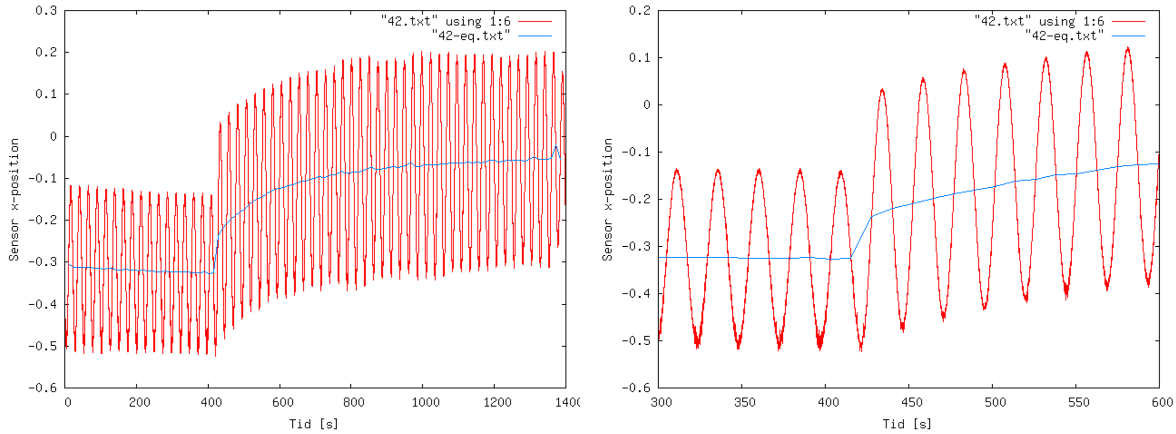
We made nearly 50 experiments in different series, which are described in more detail in the original report. Table 1 shows the experiment variables that were common for all the experiments.

The moment of inertia about the y-axis I_y was calculated using precise measurements of mass and dimension for the pendulum. The angular frequency ω_y was calculated from a gnuplot fit of several sets of experiment data. Transmission through the chamber glass t_g and reflection on the mirror r_s were measured using a laser power meter. The distance from the incidence point of the momentum-laser to the y-axis, d_y, was measured with a ruler. This value has the highest uncertainty of all experiment variables, since it was difficult to measure precisely the source-laser's point of incidence using a ruler through the glass of the vacuum chamber.

In the first series, we began by doing a few experiments the "old-fashioned way", where we did not use the photosensor. Instead, we placed a piece of paper on the wall opposite the pendulum, and made marks corresponding to the amplitude of oscillation before and after turning the

Variable	Symbol	Value	Uncertainty	Uncertainty percent	Unit
Pendulums moment of inertia about the y-axis	I_y	$5.936 \cdot 10^{-6}$	$4.4 \cdot 10^{-8}$	0.742%	$kg \cdot m^2$
Angular frequency for y-axis	ω_y	0.256	$2.3 \cdot 10^{-4}$	0.091%	s^-1
Oscillation period about the y-axis	T_y	24.5	0.02	0.091%	s
Distance from the momentum-laser incidence point to the y-axis	d_y	48	2	4.167%	mm
Transmission through the chamber glass	t_g	0.798	0.0099	1.2%	
Reflection on the mirror	r_s	0.835	0.0025	0.3%	

Tabel 1: Common experiment variables



Figur 7: Turn-on of the source-laser with equilibrium curves

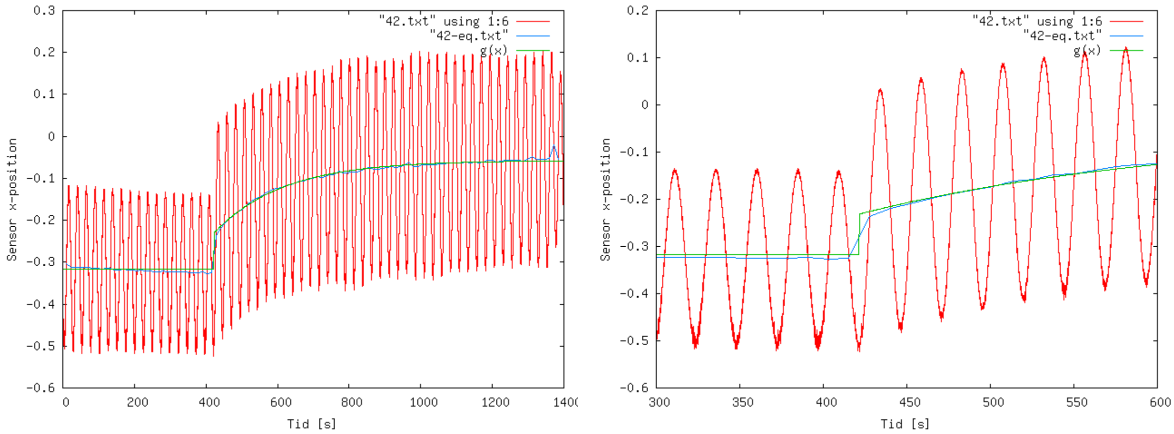
source-laser on/off. In this way, we got a feeling for the system, though the results were 2 to 4 times larger than expected from theory.

For the remaining series, we used the photosensor exclusively, calibrating it for each series. The raw data files that we recorded with LabVIEW were processed via different computer programs that we wrote, to remove noise and calculate the angular speed and acceleration of the pendulum. Using the fact that angular acceleration is zero when the pendulum is in equilibrium, equilibrium curves where generated for each recording. These curves show the pendulums equilibrium before and after turning the source-laser on/off, and were used to calculate the force exerted by the source-laser using eq 15.

We began, in the second series, with recording the pendulum's oscillation before and after turning the source-laser on/off. We also made some experiments with increasing and decreasing the amplitude of oscillation. The results were still larger than expected from theory, but significantly better - only about 40 % larger.

In series 3 and 4, we made further recordings where we aimed the source-laser at the front and back sides of both the left and right mirrors, in order to see if it made a difference, and thus determine if there was a measurable systematic error in our pendulum. Fortunately, there was no clear difference between the four sides.

Figure 7 shows the typical photosensor data from turning-on the source-laser, where the second figure shows a close-up of the turn-on time. The equilibrium curve is also shown on the figure. Here it can be clearly seen



Figur 8: Turn-on of the source-laser with fit and equilibrium curves

that the equilibrium changes instantaneously as expected with the incidence of light. However, afterwards, the equilibrium continues to change slowly in the same direction asymptotically towards a secondary equilibrium position. Due to its slow nature, we presume that this secondary change is caused by the heating of the mirrors surface and the air around it - a light-mill effect. Apparently, our vacuum was not sufficient to eliminate these effects completely.

To avoid that this secondary equilibrium change affects the results, only the first equilibrium position following the change was used to calculate the force. However, each equilibrium position is about 6 seconds apart, so the secondary force may have already had an effect on the first equilibrium position following the change. Therefore, the calculated force of light could only be seen as an upper limit.

At this stage, we suspected that the secondary equilibrium position could be the reason behind the larger than expected force measurements. Therefore, in series 5, the data recording time was lengthened to get enough data to use gnuplot to fit a simple exponential model to the secondary change. With this fit, we hoped to get a better result for the force of light. However, we found it difficult to get gnuplot to determine when the source-laser was turned on/off, and we had to set the time to halfway between the two equilibrium points bracketing the change. The precision of our result improved, but we were still getting systematically larger values for the force. Figure 8 shows the fit results for experiment 42 in experiment series 5. The fit matches well with the equilibrium curve.

Experiment series	Laser position	Laser Power [mW]	Measured F [N]	Measured F Uncertainty [%]	Theoretical F [N]	$\frac{\text{Measured } F}{\text{Theoretical } F}$
2	BH	513	$3.52 \cdot 10^{-9}$	11.00%	$2.50 \cdot 10^{-9}$	1.41
3	BH	511	$2.94 \cdot 10^{-9}$	12.00%	$2.49 \cdot 10^{-9}$	1.18
3	BV	519	$3.41 \cdot 10^{-9}$	11.70%	$2.53 \cdot 10^{-9}$	1.35
4	FV	517	$2.58 \cdot 10^{-9}$	18.30%	$2.52 \cdot 10^{-9}$	1.02
4	FH	517	$3.08 \cdot 10^{-9}$	14.30%	$2.52 \cdot 10^{-9}$	1.22
5	FH	520	$2.67 \cdot 10^{-9}$	10.30%	$2.54 \cdot 10^{-9}$	1.05
5 (fit)	FH	520	$2.93 \cdot 10^{-9}$	4.60%	$2.54 \cdot 10^{-9}$	1.15
6 (fit)	FH	514	$3.05 \cdot 10^{-9}$	4.60%	$2.51 \cdot 10^{-9}$	1.22
Average		516	$3.06 \cdot 10^{-9}$	11.80%	$2.52 \cdot 10^{-9}$	1.21 ± 0.14

Tabel 2: Average results for each experiment series

To improve the fit, a shutter mechanism was used in series 6 to place a marker in the photosensor data so we knew exactly when the source-laser was turned on/off. This helped a little with the fit results, but the results were still of same order as with the other series.

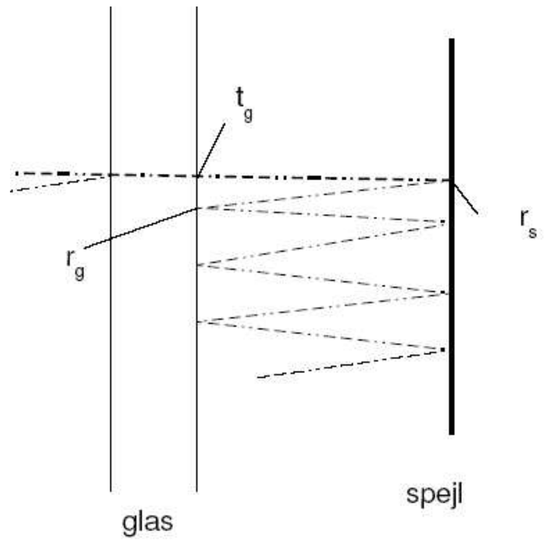
4.1 Experiment results

Table 2 shows the average results for each series with the photosensor and the combined average. The combined average is weighted by the power of the laser.

The results for the experiments are close to the theoretical. Overall, we measured a force that was about 21 (±14) percent larger than the theoretical. The uncertainty of our force measurement is about 12 %.

4.2 Multiple reflections

One possible explanation of why we systematically measured a larger force than predicted by theory may be that multiple reflections of the source-laser from the glass plate increased the force applied to the mirror.



Figur 9: Reflections between the glass and mirror (spejl)

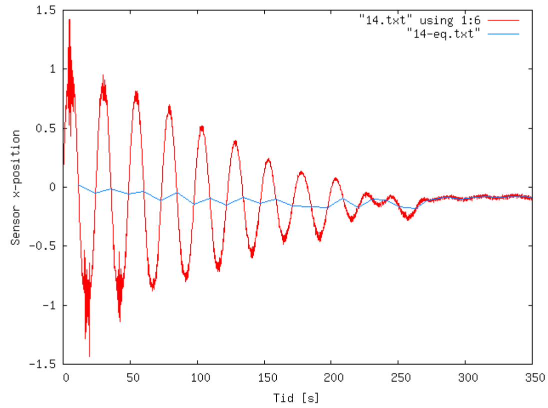
We found that the chamber glass was slightly reflective and that there were multiple visible reflections of the source-laser on the glass's surface. See figure 9.

Although, we did not initially measure the fraction of light reflected by the glass r_g, we subsequently found it to be 7.1 %. Thus, a small fraction of the light reflected by the mirror is reflected back on the mirror again. The result is a geometric series that asymptotically approaches a finite value.

We can define from eq. 16 the momentum factor for a single reflection as $f_1 = t_g(1 + r_s)$. The reflection factor for the nth reflection becomes $f_n = t_g(1 + r_s) \cdot (r_g(1 + r_s))^{n-1}$. Summing over an infinite number of reflections, we get:

$$f_\infty = \sum_{n=0}^{\infty} t_g(1 + r_s) \cdot (r_g(1 + r_s))^n = \frac{t_g(1 + r_s)}{1 - r_g(1 + r_s)} = \frac{1}{1 - r_g(1 + r_s)} f_1 \quad (17)$$

Using the values for r_g and r_s, we get a resulting momentum factor from multiple reflections that is about 15 % larger than for a single reflection. When we include this factor in the calculation of the theoretical force, the experiment results improve to within 6 % of theory, which is well within the 12 % uncertainty of our measurements. In addition, the results are no



Figur 10: Reduction in amplitude with the help of light

longer systematically larger, but are distributed both above and below the theoretical.

We now believe that the larger values we measured for the force of light were not due so much to the light-mill effect, but instead were caused by multiple reflections.

4.3 Changing the amplitude

As discussed in section 3.3, it is possible to use the momentum of light to increase or decrease the oscillation amplitude, by turning the source-laser on/off at the right times. This method was used much in the calibration of the photosensor and to zero the pendulum. One example of this is shown in figure 10.

5 Suggestions for improvement

To reduce the light-mill effect, believed responsible for the secondary equilibrium position, a tighter vacuum chamber should be used, so that the vacuum can be reduced further. A mirror that reflects light closer to 100 % may also help reduce the light-mill effect, by reducing heating from absorption.

Anti-reflective glass would reduce the effect of multiple-reflections, though these can also be taken into account in the theory.

Finally, a ruler mounted within the chamber or etchings marked on the pendulum, would enable a more precise measurement of the distance between the incident light and the y-axis of rotation. The inaccuracy of this measurement is largely responsible for the uncertainty of our measured force.

6 Conslusion

We successfully measured the force of light and shown thereby that light has momentum, despite a photon having no mass. It was encouraging to see how small a force we were able to measure ($\sim 10^{-9}$ N) and how close to the theoretical values our results were. After correcting the results, with a factor resulting from multiple reflections, our measured force was 6 percent from the theoretical, with an uncertainty of 12 percent.

The collection of data via a photosensor made it possible to see the pendulums motion in real-time, and this was a wonderful tool to see precisely what happened when the light-source was activated or deactivated.

All in all our experiment has given us some very useful results.

Litteratur

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